Border Problem
Week 1 - Days 1 & 2

Introduction:

I (Jo Boaler) first learned about this problem from Cathy Humphreys, an amazing teacher who discusses the lesson in our book: Connecting Mathematical Ideas. I really like this activity as the first introduction to algebra as students need to use algebraic expressions to describe the border of a square that they see. Students in a class will describe the border growth in different ways, they then move to writing about the growth and finally they can be helped to use variables and to create different algebraic expressions. As students describe the growth of the square in different ways, there will be many different algebraic expressions and teachers can point out that they are all equivalent. Importantly each algebraic expression will come from a visual pattern, that students can see and understand.

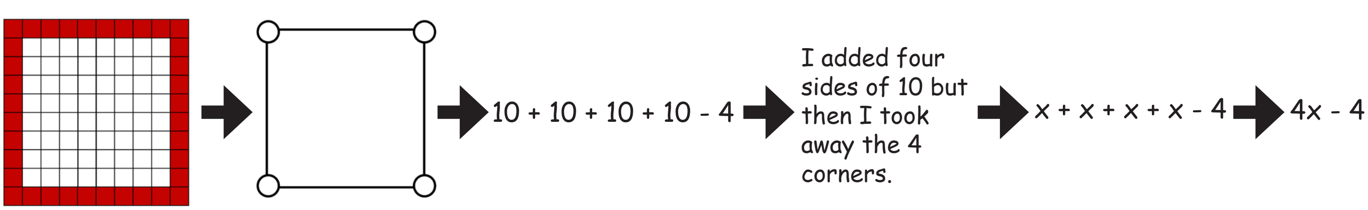
Connection to CCSS
- MP 7
- MP 8
- HSA.SSE.A.1, 2
- HSA.SSE.B
- HSF.BF.A.1

Agenda for the activity:

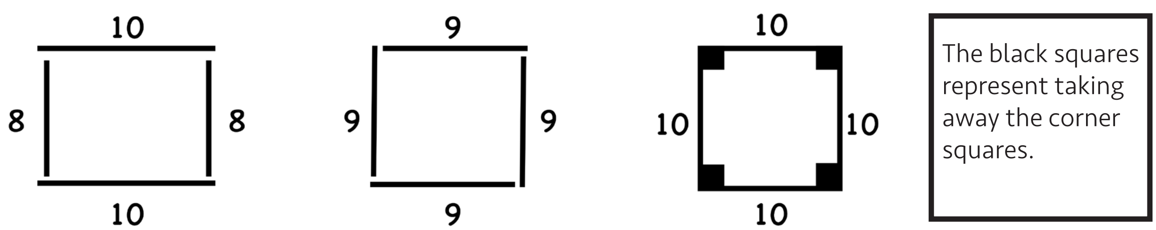
Activity	Time	Description/Prompt	Materials
10x10 Border Number Talk	15 min	How many squares are in the border of a 10x10 grid? • Show the image of the 10x10 square with a red border for a short time so students do not count the squares • Do not copy this for students as then they will count the squares • Ask how many squares are in the border of a 10x10 grid?	• Copy of the border problem 10 x 10 image to display to the class
Shrink the Grid	30 min	Choose a strategy to use on a smaller grid.	• Poster paper
Instructions for any Grid	15 min	Write a description of the strategy for finding how many squares are in the border of any sized grid.	• Paper/journal • Pencil or paper
Essence of Algebra	20 min	Translate written descriptions to algebraic expressions.	

To the teacher:

We used this activity in our summer camp (https://www.youcubed.org/resources/border-problem-video/) to introduce students to algebra. In this activity we move from thinking about the border of a 10x10 grid to a diagram of the border of any sized grid to a numeric expression to a written description to an algebraic expression to rewriting equivalent algebraic expressions. Here's a visual that illustrates this process.



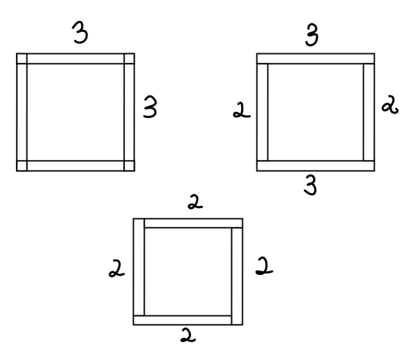
Launch:
Start by projecting a 10x10 grid like the one on page 6. We like to think about this first part as a number talk. When we post the grid, we give students time to think just like we do with a number talk. We only show the image for a few seconds so students do not count the squares. We also tell them not to count the squares. We ask them how many squares would be on the border of this 10 x 10 grid. We start by asking for a student to share their answer and then we ask for other answers. We record all of their different answers on the board without giving any indication if they are correct or incorrect. We usually collect five or more possible answers. We follow by asking students to share their way of seeing the border and their strategy for finding the number of squares in the border of a 10x10 grid. We record each strategy shared on the whiteboard with a labeled visual and a numeric expression. Here are a few of the representations we illustrate:



If there is not a variety of strategies shared, we offer additional strategies by telling the class that we saw these other strategies in another class. This gives students more strategies to think about and make sense of. For each strategy we record on the board we label them with the student's name since that is now their method.

After we discussed the strategies we asked students to apply a strategy of their choice to a 6x6 grid. We asked them to prepare a poster of their solution which includes a visual and numerical representation. When students had completed their 6x6 posters we shared them through a gallery walk and a class discussion. We discussed the different methods that groups used and the different visuals that were shared. We celebrate the fact that no poster looks anything like the others. This is a clear sign of creative maths work.

This was the end of the first lesson on the border problem. We started the next part of the lesson by doing a 3x3 grid number talk. We asked students to share their method, not the answer. We drew representations of their method accurately on the board and then waited to see what would happen. We were amazed by the understanding they had as well as the mistakes they were still making. Here are the answers the class provided for the 3x3 square. The first student said, "multiply 3 x 3 and subtract four. The second representation showed the student had missed a few corners when they said "3 + 3 + 2 + 2." The final representation was a correct one! We asked students to determine the numerical answer for each representation described. This was followed by a conversation about what was wrong with the representations since all three methods had a different answer. We asked them, "is this a problem that has one right answer or many right answers?



Our next move was to ask each team to write a description of how to use one of the strategies on any sized grid. We then asked one student from each group to write the team's description on the board. We then assigned each team another group's description and asked them to check the directions by finding the answer to any square they wanted. Students were asked to revise the directions by writing under the previous description. This way we had the previous version to refer back to because we never like to erase work. Mistakes are important and we celebrate them. This was followed by another class discussion.

Here's an example of a description that was revised.

#4

Multiply the base and height of the square. Also, because we are trying to avoid counting the small squares on each 4 ends of the square, you have to decrease your result by 4.

1st draft

Multiply the base or height of the square by 4. Next, subtract 4 because we can not double count the corners (your method works only for 4x4)

2nd draft

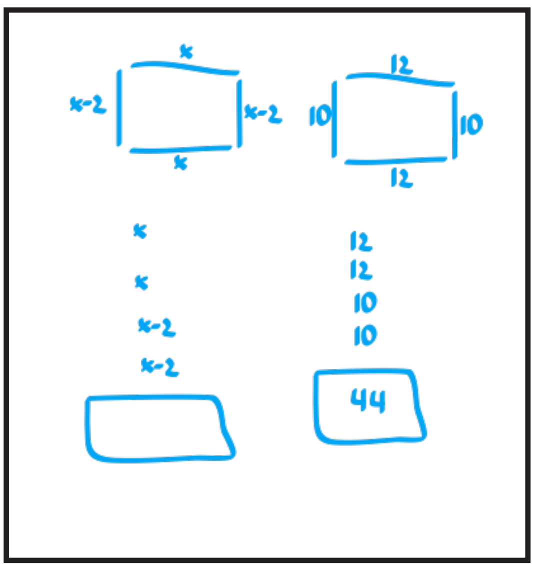
Our final move was to introduce variables by saying that instead of writing the description in words; we can use variables to write an algebraic description. This is a way to describe what you are doing more efficiently. We let students know that a variable can be any letter, J, or more commonly seen, x and y. It doesn't matter which letter you choose. We then asked students to share ideas about how they would write the description using variables. This was a challenge for our students and we had to give a lot of wait time. We were surprised by how many students were focused on the numerical answer. A second challenge was trying to help students understand that the words they were using to describe side lengths could be expressed using the same variable.

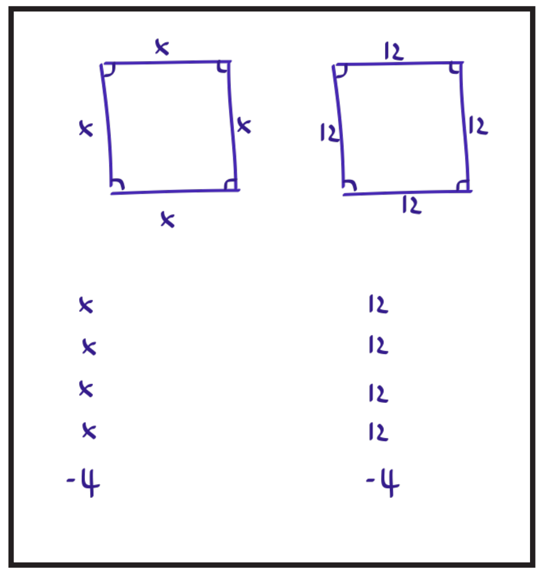
We started with the method that adds up the whole length of the top and bottom and then subtracts the two corners from the two vertical sides in order to not double count the corners. We asked how we could label the top and the bottom and a student volunteered 'x'. We labeled the top and bottom on the diagram with an x. Then we asked students what they would label the vertical sides that had the two corners subtracted. One student said 'y'. We responded that you could write it that way, however we want to try to write the description with as few variables as possible. We asked if they could think of a way of writing the vertical height minus the two corners in terms of x. Another student volunteered 'x-2'. We labeled both vertical heights 'x-2'. We then asked what we should do with these lengths. We wrote the expression under the example for the students work for the 12 by 12 square. We then worked through another method (x(4)−4).

We acknowledged that this new way of thinking about the problem was new and hard. We described to the students that what they had been doing was building and improving upon other people's thinking. We described this as a type of 'design thinking' that is used a lot in the Silicon Valley. Somebody suggests an idea and others revise and build on that idea. We let them know that this is an important aspect of what we do in this mathematics class.

We then moved the conversation to rewriting equivalent expressions by asking, "Is 4x – 4 the same or different from x+x+(x-2)+(x-2)?"

We started by organizing the information for the students in the following manner on the board for the two different methods.



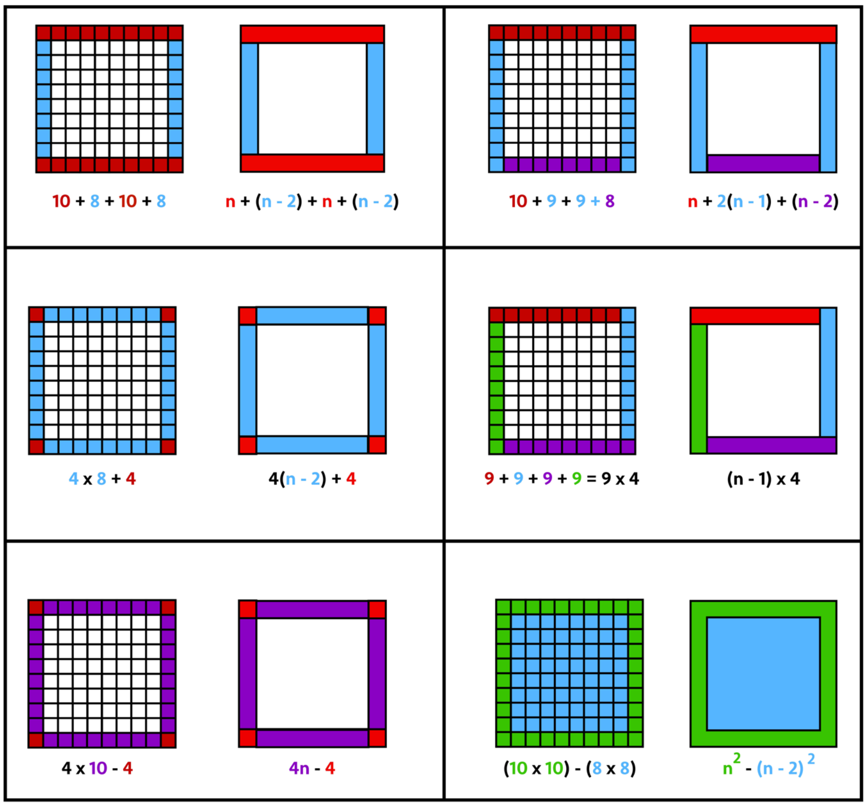


We asked students, what would happen if you added the variables together for method 1 and what would happen if you add the variables from method 2. We let them know they might not know how to do it, but they could just try it. 'Remember, if you try it and it is wrong that can be more synapses firing!' Before getting started, one student asked what 'x' was. He wanted to know what number it was. We let students know that the variable could represent any number. We asked students to write down what 'x' was representing in words, in their journals.

Growth Mindset Message: 'We are all about showing you an idea and struggling with it. Remember the more you struggle with an idea, the more your brain will grow. We could give you things to do in this class that you would get all correct, but it really wouldn't help you. It might feel good, but it wouldn't help you. What you really want to be doing is struggling with ideas.'

After giving students time to talk with their team to ask questions and share ideas about our question, we asked a person from each team to share what the group discussed. Through the discussion we got to the place that students were convinced that all the different expressions can we written as 4x−4.

Here's a visual of the different solutions we saw. We don't recommend that you show this to students. Let them come up with their solutions.



Additional Resource:

Boaler, J., & Humphreys, C. (2005). Connecting mathematical ideas: Middle school video cases to support teaching and learning. Heinemann Educational Books.

Extensions for the activity:

- Sketch a diagram and use the algebraic expression to find the number of squares in the border of a 75x75 grid.
- Give teams another geometric pattern to think about how they see it, how the shape changes when the dimensions change, and to represent what they see algebraically.